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2000 J. Phys. A: Math. Gen. 33 8305

(http://iopscience.iop.org/0305-4470/33/46/311)

J. Phys. A: Math. Gen. **33** (2000) 8305–8318. Printed in the UK PII: S0305-4470(00)16049-4

Quantum mechanics of smeared particles

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Received 27 July 2000, in final form 6 September 2000

Abstract. We propose quantum mechanics of smeared particles that account for the delocalization of a particle defined via its Compton wavelength. The Hilbert space representation theory of such quantum mechanics is presented and its invariance under spatial translations and rotations is examined. The quantum mechanics of smeared particles is then applied to two paradigm examples, namely, the smeared harmonic oscillator and the Yukawa potential. In the second example, we theoretically predict the phenomenological coupling constant of the ω meson, which mediates the short range and repulsive nucleon force, as well as the repulsive core radius.

1. Introduction

The representation of a particle as an idealized point has long been used in physics. In fact, this representation is central to classical mechanics and serves us well even in quantum mechanics. In this paper we adopt a viewpoint in which the nonlocality or the smearing of a particle is taken into consideration thereby treating the point particle as a *smeared* particle. Such a treatment becomes important and necessary when the confines of the quantum system in which the particle is placed become comparable to the smearing of the particle. The smearing of a particle is quantified via its Compton wavelength which can be defined as the lower limit on how well a particle can be localized. In nonrelativistic quantum mechanics, the lower limit is zero since we admit position eigenkets $|x\rangle$. But in reality, as we try to locate the particle with greater accuracy, we use more energetic probes, say photons to be specific. To locate a particle to some Δx we need a photon of momentum

$$\Delta p \approx \frac{\hbar}{\Delta x}. \tag{1.1}$$

The corresponding energy of the photon is

$$\Delta E \approx \frac{\hbar c}{\Delta x}. \tag{1.2}$$

If this energy exceeds twice the rest energy of the particle, relativity allows the production of a particle–antiparticle pair in the measurement process. So we demand

$$\frac{\hbar c}{\Delta x} \leqslant 2mc^2 \qquad \text{or} \qquad \Delta x \geqslant \frac{\hbar}{2mc} \approx \frac{\hbar}{mc}. \tag{1.3}$$

Any attempt to further localize the particle will lead to pair creation and we will have three (or more) particles instead of the one we started to locate. Therefore, the Compton wavelength of a particle measures the distance over which quantum effects can persist. The point-particle approximation used in nonrelativistic quantum mechanics suffices to describe the dynamics since the confines of the quantum systems under consideration are much larger than the

0305-4470/00/468305+14$30.00

smearing of the confined particles. For example, in the analysis of the hydrogen atom, the smearing of the electron is α times smaller than the size of the atom a_0:

$$\frac{\hbar/mc}{a_0} = \alpha \approx \frac{1}{137}. \tag{1.4}$$

Thus, in the case of the hydrogen atom and in general, for the quantum theory of atoms, the quantum mechanics of point particles gives an accurate description.

In this paper we develop the Hilbert space representation theory of the quantum mechanics of smeared particles. We use this representation to analyse two paradigm examples: the smeared harmonic oscillator and the Yukawa potential. In the second example, the quantum mechanics of smeared particles enables us to predict the phenomenological coupling constant of the ω meson as well as the radius of the repulsive nucleon core.

2. Quantum mechanics of smeared particles

We have established the necessity for taking into consideration the nonzero smearing of a particle. In order to incorporate the smearing of a particle into our dynamics we introduce the following representation for position and momentum in one dimension in units where $\hbar = c = 1$. For position space,

$$\begin{aligned} &X_{\rm s} = (X{\rm e}^{-P^2/m^2}) \to (x{\rm e}^{-P^2/m^2}) \\ &P \to -{\rm i}\frac{\rm d}{{\rm d}x} \\ &[X_{\rm s}, P] = {\rm i}{\rm e}^{-P^2/m^2} \end{aligned} \tag{2.1}$$

and for momentum space,

$$\begin{aligned} &X_{\rm s} = {\rm e}^{-P^2/2m^2} X {\rm e}^{-P^2/2m^2} \to {\rm i}{\rm e}^{-P^2/2m^2}\frac{\rm d}{{\rm d}p}{\rm e}^{-P^2/2m^2} \\ &P \to p \\ &[X_{\rm s}, P] = {\rm i}{\rm e}^{-p^2/m^2} \end{aligned} \tag{2.2}$$

where $(AB) \equiv (AB + BA)/2$. Symmetrization has also been employed in the momentum space representation in order to preserve the Hermiticity of the noncommuting smeared position operator $X_{\rm s}$. In contradistinction to the quantum mechanics of point particles where the position operator has a smooth coordinate representation consisting of a sequence of points, the smeared position operator is convolved with a Gaussian in momentum space which has as its width the Compton wavelength $1/m$. The convolution with the Gaussian has the effect of smearing out these points, and in the limit as the Compton wavelength vanishes, we recover the standard operator assignments of ordinary quantum mechanics. For simplicity, we consider the effect of the smeared position operator $X_{\rm s}$ on an acceptable wavefunction in position space, that is, one which is square integrable and has the right behaviour at infinity:

$$\begin{aligned} X_{\rm s}\psi(x) &= (x{\rm e}^{-P^2/m^2})\psi(x) \\ &= \frac{m}{4\sqrt{\pi}}\left[\int_{-\infty}^{\infty}{\rm d}\lambda\, x{\rm e}^{{\rm i}P\lambda - m^2\lambda^2/4}\psi(x) + \int_{-\infty}^{\infty}{\rm d}\lambda\,{\rm e}^{{\rm i}P\lambda - m^2\lambda^2/4}[x\psi(x)]\right] \\ &= \frac{m}{2\sqrt{\pi}}\int_{-\infty}^{\infty}{\rm d}\lambda\left(x + \frac{\lambda}{2}\right)\psi(x+\lambda){\rm e}^{-m^2\lambda^2/4}. \end{aligned} \tag{2.3}$$

The translation of $\psi(x)$ by λ and the subsequent integration over all possible values of λ weighted by a Gaussian measure has the effect of smearing out the position. As an example,

consider the effect of the smeared position operator on the usual position coordinate x. After evaluating the integral in equation (2.3) we obtain

$$X_s x = x^2 + \frac{1}{m^2}. \tag{2.4}$$

We observe that the smeared position operator multiplies the usual position by x (as does the ordinary position operator X) but there is an additional constant term which depends on the Compton wavelength $1/m$. The curve defined by Xx is the parabola x^2 whereas the curve defined by $X_s x$ is the shifted parabola $x^2 + 1/m^2$. The region in between these two curves is the indeterminate or smeared region. As the Compton wavelength goes to zero, the two curves become identical, and the smearing vanishes. Physically, we can interpret the smearing of the position coordinate by saying that the smearing operator encodes the Compton wavelength of a particle which in turn represents the lower limit on how well the particle can be localized. The commutation relation obeyed by X_s and P is manifestly noncanonical and does not depend on the representation. A direct consequence of this commutation relation is the uncertainty relation

$$\Delta X_s \Delta P \geqslant \tfrac{1}{2}|\langle e^{-P^2/m^2}\rangle|. \tag{2.5}$$

Now, for any two observables A and B which satisfy $[A, B]|\psi\rangle = 0$ for some nontrivial $|\psi\rangle$, with uncertainties ΔA and ΔB such that $|\Delta A/\langle A\rangle| \ll 1$ and $|\Delta B/\langle B\rangle| \ll 1$, we have the relation

$$\Delta((AB)) = |\langle A\rangle|\Delta B + |\langle B\rangle|\Delta A \tag{2.6}$$

where again $(AB) \equiv (AB + BA)/2$. If the above conditions are satisfied for any two observables, this relation follows directly from the theory of uncertainty propagation. For the special case $[A, B] = 0$ we say that A and B are compatible variables. We observe that whenever simultaneous eigenkets exist

$$\begin{aligned}\langle AB\rangle &= \int da\, db\, P(ab)ab = \int da\, db\, P(a)P(b)ab \\ &= \langle A\rangle\langle B\rangle\end{aligned} \tag{2.7}$$

where $P(ab) = |\langle ab|\psi\rangle|^2$ and the proof of equation (2.6) follows. In our case,

$$[X, e^{-P^2/m^2}]|\psi\rangle = 0 \qquad \text{only if} \quad |\psi\rangle = \text{constant}. \tag{2.8}$$

Hence, there exists at least one nontrivial simultaneous eigenket for which $[X, e^{-P^2/m^2}]$ has a zero eigenvalue. We can always choose this eigenket to establish the validity of equation (2.6) for our operators X and e^{-P^2/m^2} along the lines shown above. As a consequence, we obtain the modified uncertainty principle (reinserting $\hbar$ for clarity)

$$\Delta X \Delta P \geqslant \frac{\hbar}{2} + \frac{2|\langle X\rangle||\langle P\rangle|}{m^2}(\Delta P)^2. \tag{2.9}$$

The uncertainty product goes up because of the smearing we have introduced in the position. Consequently, there exists a minimal uncertainty in position given by

$$\Delta X_0 = \frac{2}{m}\sqrt{|\langle X\rangle||\langle P\rangle|\hbar}. \tag{2.10}$$

The existence of minimal uncertainties and their consequences for structure were first examined by Kempf, albeit in a different context [1, 2]. If we view the uncertainty product as a measure of the cell volume of phase space we observe that quantized phase space acquires an added smearing and the cell volume no longer has a uniform value equal to the Planck constant. Thus, by introducing these representations for the smeared position we are able to quantify and characterize the delocalization of a particle. We now proceed to formulate the Hilbert space representation theory of these operators.

3. Hilbert space representation

The smeared position operator X_s and the momentum operator P satisfy the uncertainty relation equation (2.5). This relation does not imply a minimal uncertainty in the smeared position or the momentum. As a consequence, the eigenstates of the self-adjoint smeared position and momentum operators can be approximated to arbitrary precision by sequences $|\psi_n\rangle$ of physical states of increasing localization in position or momentum space:

$$\lim_{n\to\infty} \Delta X_{s_{|\psi_n\rangle}} = 0 \qquad \text{or} \qquad \lim_{n\to\infty} \Delta P_{|\psi_n\rangle} = 0. \tag{3.1}$$

Hence, the smeared position and momentum operators admit a continuous position or momentum space representation in the Hilbert space. Since the momentum operator is identical to the one used in ordinary quantum mechanics it has the usual orthogonal plane wave eigenstates. The eigenvalue problem of the smeared position operator

$$X_s\psi = \lambda\psi \tag{3.2}$$

can be written in the momentum basis (which we choose for convenience) as

$$e^{-p^2/2m^2}\frac{d}{dp}(e^{-p^2/2m^2}\psi) = -i\lambda\psi. \tag{3.3}$$

Defining the function $\phi = e^{-p^2/m^2}\psi$ and introducing the measure transformation $dr = e^{p^2/m^2}dp$ we obtain the eigensolutions

$$\psi(p) = \frac{1}{\sqrt{2\pi}}e^{p^2/2m^2+i\lambda r} \tag{3.4}$$

where freedom in scale has been used to normalize the solution. The eigenfunctions are orthogonal with respect to the transformed measure $L^2(e^{-p^2/m^2}dr)$ because

$$\langle\psi_\lambda(p)|\psi_{\lambda'}(p)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\lambda-\lambda')r}\,dr = \delta(\lambda-\lambda'). \tag{3.5}$$

The inner product $\langle\psi_\lambda(p)|\psi_{\lambda'}(p)\rangle$ is divergent in the space $L^2(dp)$ but is equal to the Dirac delta function in the space $L^2(e^{-p^2/m^2}dr)$. As p ranges from $-\infty$ to ∞, the volume element dp, under the measure transformation, is squeezed into a Gaussian width times the line element dr, and consequently the orthogonality of the smeared position eigenstates is preserved. We note that had we tried to construct the formal position eigenstates (eigenstates of X) we would have had to sacrifice orthogonality due to the appearance of the minimal uncertainty in position. The eigenfunctions of the smeared position operator in the position representation will be Fourier transforms of the eigensolutions in the momentum representation since the Fourier transform of an L^2 function will be an L^2 function in the same measure.

4. Translational and rotational invariance

We will now examine the behaviour of the quantum mechanics of smeared particles under spatial translations and rotations and solve the eigenvalue problem of smeared angular momentum.

4.1. Translational invariance

Under a translation of the coordinate $x \to x + \epsilon$ we have the smeared translation

$$\begin{aligned} &\langle X_s\rangle \to \langle X_s\rangle + \epsilon\langle e^{-P^2/m^2}\rangle \\ &\langle P\rangle \to \langle P\rangle. \end{aligned} \tag{4.1}$$

In the passive transformation picture

$$\begin{aligned} T^{\dagger}(\epsilon)X_{\mathrm{s}}T(\epsilon) &= X_{\mathrm{s}} + \epsilon \mathrm{e}^{-P^2/m^2} \\ T^{\dagger}(\epsilon)PT(\epsilon) &= P \end{aligned} \tag{4.2}$$

where $T(\epsilon)$ is the translation operator which translates the state $|\psi\rangle$. Expanding $T(\epsilon)$ to first order and feeding into equation (4.2) we obtain

$$[X_{\mathrm{s}}, G] = \mathrm{i}\mathrm{e}^{-P^2/m^2} \tag{4.3}$$

where G is the generator of infinitesimal translations. Thus, the momentum is still the generator of smeared spatial translations. Since these are the same generators as found in ordinary quantum mechanics, we can conclude by similar reasoning and by Ehrenfest's theorem that smeared space translational invariance will ensure the time independence of the momentum.

4.2. Rotational invariance

Let us denote the operator that rotates two-dimensional vectors by $R(\phi_0\hat{k})$ for a rotation by ϕ_0 about the z-axis. Let $U[R]$ be the operator associated with this rotation. For an infinitesimal rotation $\epsilon_z\hat{k}$ we set

$$U[R] = I - \mathrm{i}\epsilon_z L_{\mathrm{s}_z} \tag{4.4}$$

where L_{s_z} is the generator of smeared rotations. We can determine $L_{\mathrm{s}_z} = X_{\mathrm{s}}P_y - Y_{\mathrm{s}}P_x$ by feeding this $U[R]$ into the passive transformation equations for an infinitesimal rotation:

$$U^{\dagger}[R]X_{\mathrm{s}}U[R] = X_{\mathrm{s}} - Y_{\mathrm{s}}\epsilon_z \tag{4.5}$$

and so on. L_{s_z} is conserved in a problem with rotational invariance if

$$U^{\dagger}[R]H(X_{\mathrm{s}}, P_x; Y_{\mathrm{s}}, P_y)U[R] = H(X_{\mathrm{s}}, P_x; Y_{\mathrm{s}}, P_y). \tag{4.6}$$

It follows (by choosing an infinitesimal rotation) that

$$[L_{\mathrm{s}_z}, H] = 0 \qquad \text{or} \qquad \langle \dot{L}_{\mathrm{s}_z} \rangle = 0 \tag{4.7}$$

by Ehrenfest's theorem.

4.3. The eigenvalue problem of L_{s_z}

In the momentum basis the two-dimensional smeared angular momentum operator can be written as

$$L_{\mathrm{s}_z} \to \mathrm{e}^{-p^2/2m^2}\left(\mathrm{i}\frac{\partial}{\partial_{p_x}}\mathrm{e}^{-p^2/2m^2}p_y - \mathrm{i}\frac{\partial}{\partial_{p_y}}\mathrm{e}^{-p^2/2m^2}p_x\right) \tag{4.8}$$

where $p^2 = p_x^2 + p_y^2$. This is the correct generalization of the smeared position operator to higher dimensions (in this case two) as can be seen by letting X_{s} act on a wavefunction in two dimensions. We can further simplify the derivatives in L_{s_z} and switch to polar coordinates to obtain

$$L_{\mathrm{s}_z} \to -\mathrm{i}\mathrm{e}^{-p^2/2m^2}\frac{\partial}{\partial_{p_\phi}}\mathrm{e}^{-p^2/2m^2}. \tag{4.9}$$

The eigenvalue problem of L_{s_z},

$$L_{\mathrm{s}_z}\psi(p_\rho, p_\phi) = l_{\mathrm{s}_z}\psi(p_\rho, p_\phi) \tag{4.10}$$

can be written in the momentum basis as

$$-\mathrm{i}\mathrm{e}^{-p^2/2m^2}\frac{\partial}{\partial_{p_\phi}}(\psi\mathrm{e}^{-p^2/2m^2}) = l_{\mathrm{s}_z}\psi. \tag{4.11}$$

Defining $\phi = \psi e^{-p^2/m^2}$ and using the transformed measure

$$d p_\phi = \frac{1}{2\pi}\left[\frac{\sqrt{\pi}m}{2i}\,\mathrm{erf}(2\pi i)\right] e^{-p_\phi^2/m^2}\, dr \tag{4.12}$$

we arrive at

$$\psi(p_\rho, p_\phi) \sim \exp(i l_{s_z} e^{p_\rho^2/m^2} r + p^2/2m^2) \tag{4.13}$$

where the numerical factor in the measure transformation has been chosen so that as p_ϕ ranges from 0 to 2π, r also ranges from 0 to 2π. The eigenfunctions are orthogonal with respect to the transformed measure $L^2(e^{-p_\phi^2/m^2} p_\rho d p_\rho\, dr)$ where the numerical factor has been suppressed. We observe that l_{s_z} seems to be arbitrary and even complex since the range of r is restricted. The fact that complex eigenvalues enter the solution signals that we are overlooking the Hermiticity constraint. Imposing this condition we have

$$\langle\psi_1|L_{s_z}|\psi_2\rangle = \langle\psi_2|L_{s_z}|\psi_1\rangle^* \tag{4.14}$$

which becomes in the momentum basis

$$\int_0^\infty \int_0^{2\pi} \phi_1^*\left(-i\frac{\partial}{\partial_{p_\phi}}\right)\phi_2 p_\rho\, d p_\rho\, d p_\phi = \left[\int_0^\infty \int_0^{2\pi} \phi_2^*\left(-i\frac{\partial}{\partial_{p_\phi}}\right)\phi_1 p_\rho\, d p_\rho\, d p_\phi\right]^* \tag{4.15}$$

where $\phi = \psi e^{-p^2/2m^2}$. If this requirement is to be satisfied by all ϕ_1 and ϕ_2, one can show (by integrating by parts) that it is enough if each $\phi(p_\rho, p_\phi)$ obeys

$$\phi(p_\rho, 0) = \phi(p_\rho, 2\pi). \tag{4.16}$$

If we impose this constraint on the L_{s_z} eigenfunctions we find that the eigenvalues l_{s_z} have to obey the following relation:

$$l_{s_z} = e^{-p_\rho^2/m^2} k \tag{4.17}$$

where k is an integer. The smeared angular momentum is equal to an integral multiple of $\hbar$ times a smearing factor. This is an example of smeared quantization and, as the Compton wavelength vanishes, we regain the usual relation for ordinary quantized angular momentum. Having established the Hilbert space representation theory for smeared particles, we now apply it to our first paradigm example, namely, the smeared harmonic oscillator.

5. Smeared harmonic oscillator

Before we study the quantum mechanical smeared harmonic oscillator let us understand the classical analogue of such an oscillator. Classically, we can model a smeared particle as a point mass connected to a nonlinear spring of stiffness constant, say k_1. When this spring–mass system is connected to another linear spring of stiffness constant, say k_2 we essentially have a classical, one-dimensional, smeared oscillator. When the wavelength of oscillation is small compared to the size of the smeared particle (in this case the length of the nonlinear spring of stiffness constant k_1) the oscillator will exhibit harmonic behaviour since the small oscillations do not disturb the configuration of the smeared particle. As the wavelength of oscillation becomes comparable to the size of the smeared particle, anharmonic vibrations set in. Again, as the wavelength of oscillation becomes much larger than the size of the smeared particle, the point-particle approximation becomes tenable and harmonic vibrations are recovered. We would expect the quantum version of the smeared oscillator to exhibit similar behaviour albeit with quantized energy levels. In the first regime, when the wavelength of oscillation is small compared to the size of the particle, since small oscillations do not disturb the configuration of the smeared particle to any appreciable extent, we will obtain the usual quantized energy levels

of the simple harmonic oscillator. It is in the second and third regimes where we would need to apply the quantum mechanics of smeared particles. The Hamiltonian for a one-dimensional smeared harmonic oscillator can be written as

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X_s^2. \tag{5.1}$$

Introducing the operator representation for the smeared position and momentum in the momentum basis and simplifying terms, we obtain

$$\frac{1}{2}m\omega^2\left[\frac{\mathrm{d}^2\phi}{\mathrm{d}p^2} - \left(\frac{p^2}{m^4} - \frac{1}{m^2}\right)\phi\right] = \left(\frac{p^2}{2m} - E\right)\mathrm{e}^{2p^2/m^2}\phi \tag{5.2}$$

where $\phi = \mathrm{e}^{-p^2/m^2}\psi$, $H\psi = E\psi$, and ϕ lies in $L^2(\mathrm{d}p)$. When the wavelength of oscillation (the confines) is large compared to the size of the smeared particle, $p^2/m^2 \ll 1$, in which case we can approximate $\mathrm{e}^{2p^2/m^2} \approx 1 + 2p^2/m^2$. In this approximation equation (5.2) can be rewritten as

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}p^2} + 2m\left(\tilde{E} - \frac{1}{2}m\Omega^2\right)\phi = 0 \tag{5.3}$$

where

$$2m\tilde{E} = \frac{2E}{m\omega^2} + \frac{1}{m^2} \tag{5.4}$$

$$m^2\Omega^2 = \frac{-4E}{m^3\omega^2} + \frac{1}{m^4} + \frac{1}{m^2\omega^2}. \tag{5.5}$$

This is simply the differential equation for a simple harmonic oscillator in terms of the dummy energy $\tilde{E}$ and frequency Ω. For well behaved solutions we require the quantization condition

$$\tilde{E}_n = (n + \tfrac{1}{2})\Omega \qquad n = 0, 1, 2, \ldots. \tag{5.6}$$

Re-expressing this relation in terms of the physical energy E and frequency ω and retaining terms up to $\mathrm{o}(\hbar^2)$, we obtain

$$E_n = \left(n + \frac{1}{2}\right)\omega - \frac{\omega^2}{2m} \qquad n = 0, 1, 2, \ldots. \tag{5.7}$$

As we would expect, the smeared particle exhibits harmonic behaviour when the wavelength of oscillation is large compared to the size of the particle. In this approximation, the eigenvalue spectrum of the smeared harmonic oscillator is equivalent to the spectrum of a displaced simple harmonic oscillator. The shift in the energy spectrum can be understood by observing that in the classical spring–mass model, the smeared particle (the nonlinear spring) would undergo compression due to the oscillations of the linear spring, thereby displacing the equilibrium position. The quantum counterpart exhibits the same behaviour and when $\omega \ll m$ in equation (5.7), that is, when the point-particle approximation becomes tenable we obtain the eigenspectrum of the simple harmonic oscillator. In the classical analogue this would mean that, at sufficiently large oscillation wavelengths, the compression of the nonlinear spring becomes insignificant. Retaining terms up to $\mathrm{o}(\hbar^2)$, the eigenfunctions of the harmonic oscillator in this approximation are given by

$$\psi(p) \sim \mathrm{e}^{p^2/m^2(1-\frac{m}{2\omega})}H_n\left[\sqrt{(m\omega)^{-1}}\,p\right] \tag{5.8}$$

where H_n are the Hermite polynomials. Since ψ lies in $L^2(\mathrm{e}^{-2p^2/m^2}\,\mathrm{d}p)$, the eigenfunctions will be normalizable. By inserting these approximate solutions into the exact differential equation (5.2) we find that they do not differ by derivative terms and hence they are close in some sense to the exact solutions.

If we include higher values of momenta in our approximation and write $e^{2p^2/m^2} \approx 1 + 2p^2/m^2 + 2p^4/m^4$, we obtain the differential equation

$$\frac{d^2\phi}{dp^2} + 2m\left(\frac{\alpha}{2m} - \frac{\beta}{2m}p^2 - \frac{\gamma}{2m}p^4\right)\phi = 0 \tag{5.9}$$

where

$$\alpha = \frac{2E}{m\omega^2} + \frac{1}{m^2} \tag{5.10}$$

$$\beta = \frac{-4E}{m^3\omega^2} + \frac{1}{m^4} + \frac{1}{m^2\omega^2} \tag{5.11}$$

$$\gamma = \frac{2}{m^4\omega^2} - \frac{4E}{m^5\omega^2}. \tag{5.12}$$

This is the differential equation for an anharmonic oscillator. As we would expect when higher momentum values become important or, equivalently, as the wavelength of oscillation becomes comparable to the size of the smeared particle, anharmonic vibrations set in. We can compute the eigenspectrum of the anharmonic oscillator using perturbation theory. We note that the perturbation expansion breaks down for some large enough n. Retaining terms up to $o(\hbar^2)$ the eigenspectrum is found to be

$$E_n = \left(n + \frac{1}{2}\right)\omega - \frac{\omega^2}{2m} + \frac{3\omega^2}{4m}(1 + 2n + 2n^2) \qquad n = 0, 1, 2, \ldots. \tag{5.13}$$

Figure 1 shows a plot of the first two anharmonic oscillator eigenfunctions. For comparison the first two harmonic oscillator eigenfunctions are also shown. The anharmonic oscillator eigenfunctions have a steeper slope because the particle is placed in a stronger potential as compared to the harmonic oscillator potential. If we include even higher values of momenta in our approximation we find that the anharmonicity increases and in the limit of large quantum numbers our quantum descriptions pass smoothly to their classical counterparts. Therefore, the quantum mechanics of smeared particles provides a description of the smeared harmonic oscillator which augments our classical intuition. Such a description could be useful when we study harmonic excitations of quasiparticles which cannot be localized to arbitrary precision. The quantum mechanics of smeared particles can also be used to describe compound particles such as baryons or mesons in situations where their nonzero size matters but the details of the internal structure do not contribute. One such situation is the description of the nucleon–nucleon interaction at very short distances which we will now proceed to examine.

6. The Yukawa potential

At present the physics of the nucleon–nucleon interaction can be divided into three major regions [3]:

(1) the *long-distance* region $r \geqslant 2$ fm $\approx 1.5m_\pi^{-1}$ where one-pion exchange dominates and the quantitative behaviour of the potential is very well established;
(2) the *intermediate* region $0.8 \leqslant r \leqslant 2$ fm where the dynamical contributions from two-pion exchange (effective boson exchange) compete with or exceed the one-pion exchange potential;
(3) the *inner* region $r \leqslant 0.8$ fm has complicated dynamics not readily accessible to a quantitative theoretical description. This region is expected to be influenced by heavy mesons and/or by quark/gluon degrees of freedom. It is usually approached in a phenomenological way.

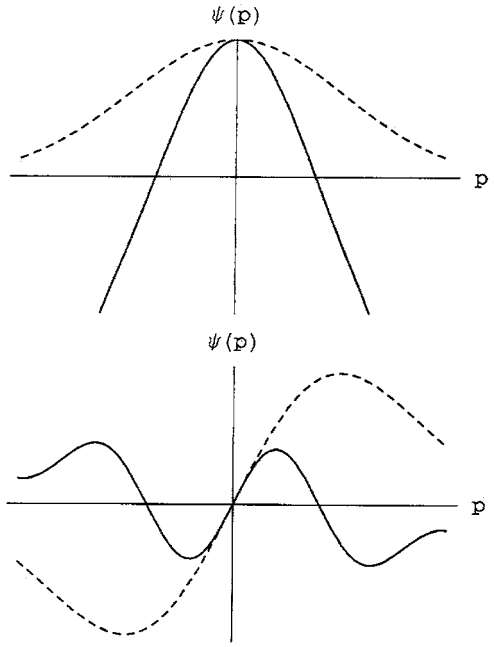

Figure 1. The first two eigenfunctions of the anharmonic oscillator (solid curves). For comparison the first two eigenfunctions of the anharmonic oscillator are also shown (dashed curves). The anharmonic oscillator eigenfunctions show a steeper slope because the particle experiences a stronger potential.

Moreover, the inner region contains a repulsive hard core of radius 0.6 fm which was first proposed by Jastrow in 1951 in order to fit nucleon–nucleon scattering data [4]. The presence of a repulsive nucleon core is necessary to explain the saturation of nuclear forces and its theoretical origin is unknown. This short range and repulsive nucleon force is believed to be mediated by an ω meson of mass 782 MeV and the intermediate range, attractive nucleon force is mediated by a σ meson (effective boson) of mass 550 MeV [5]. Once the masses are fixed, the coupling constants which measure the strength of the coupling between a meson and a baryon are chosen to reproduce nucleon–nucleon scattering phase shifts and deuteron properties. These phenomenological coupling constants [5] are found to be $g_\omega^2/4\pi = 10.83$ and $g_\sigma^2/4\pi = 7.303$. It is our objective to theoretically determine the radius of the repulsive nucleon core and to reproduce the phenomenological ω meson coupling constant using the quantum mechanics of smeared particles, which becomes relevant to the dynamics in the inner region due to the delocalization of the nucleon.

In order to reproduce consistent results we will focus attention on the bound state nucleon–nucleon interaction, namely, the deuteron. The deuterium nucleus ($A = 2, Z = N = 1$) is a bound state of the neutron–proton system, into which it may be disintegrated by irradiation with γ rays of energy above the binding energy [6] of 2.226 MeV. The ground state of the deuteron is a triplet S state and it has no excited states. The force between the proton and the neutron can be described in good approximation by a potential energy function of the form

$$V(r) = -V_0 \frac{\mathrm{e}^{-r/r_0}}{r/r_0}. \tag{6.1}$$

This is the well known Yukawa potential and is central to the mesonic theory of nuclear forces. The range of the force r_0 is equal to $1/\mu$, where μ is the mass of the associated meson and the strength V_0, or depth of the potential well, is connected with the strength of the coupling between the meson and the nucleon field. In the centre-of-mass coordinates the Hamiltonian

for the S state of the deuteron is

$$H = \frac{p^2}{2m} + V(r) \tag{6.2}$$

where m is the reduced mass of the deuteron and r determines the neutron–proton separation. For ease of comparison with the quantum mechanics of smeared particles in which the momentum basis is more convenient, we can transcribe the Hamiltonian to the momentum basis by virtue of the exchange transformation

$$r \to pr_0^2 \qquad \text{and} \qquad p \to -r/r_0^2. \tag{6.3}$$

The exchange transformation is a canonical transformation and does not affect the dynamics [7]. The Hamiltonian in the momentum basis is

$$H = \frac{r^2}{2mr_0^4} + V(p) \tag{6.4}$$

where $\boldsymbol{r} \to \mathrm{i}\nabla_{\boldsymbol{p}}$ is the position operator and $V(p) = -V_0\mathrm{e}^{-pr_0}/pr_0$. The binding energy $E_0 = -2.226$ MeV can be estimated by means of the variational principle using the simple trial wavefunction

$$\psi(p) = \mathrm{e}^{-\alpha pr_0} \tag{6.5}$$

in which we treat α as a variable parameter. Our choice of the trial wavefunction is motivated by the fact that we expect the ground state wavefunction to have no angular momentum, no nodes, and for $p\psi(p)$ to vanish as $p \to \infty$, as required for bound states. The variational method determines the energy as

$$E = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}. \tag{6.6}$$

The energy E serves as an upper bound on the ground state energy E_0. If we substitute $E_0 = -2.226$ MeV for E we can perform an approximate calculation of the relation between V_0 and r_0 (range–depth relation) that must hold if the potential function $V(p)$ is to give the value $E_0 = -2.226$ MeV for the binding energy. Figure 2 shows a plot of the range–depth relation for the Yukawa potential (deuteron) as determine by this method. By comparing the values of V_0 for various values of r_0 with the results of an exact calculation using numerical integration we are able to estimate the accuracy of our approximate result. The approximate result is within a few per cent of the exact result and the error decreases with increasing r_0 [6]. Therefore, our choice of the trial wavefunction is justified.

Let us now analyse the same potential problem using the quantum mechanics of smeared particles. In the momentum basis the smeared Hamiltonian for the S state of the deuteron is

$$H = \frac{r_{\mathrm{s}}^2}{2mr_0^4} + V(p) \tag{6.7}$$

where

$$\boldsymbol{r}_{\mathrm{s}} \to \mathrm{i}\mathrm{e}^{-p^2/2m^2}\nabla_{\boldsymbol{p}}\mathrm{e}^{-p^2/2m^2} \tag{6.8}$$

is the smeared position operator which now determines the neutron–proton separation. Figure 3 shows a plot of the S-state eigenfunctions as a function of momentum for $r_0 = 1.43$ fm, which correspond to a π meson of mass 139.6 MeV, and for $r_0 = 0.3596$ fm, which correspond to a σ meson of mass 550 MeV. The eigenfunctions obtained from ordinary quantum mechanics are also shown for comparison. The eigenfunctions obtained from the quantum mechanics of smeared particles are pushed out in comparison to the usual eigenfunctions, implying that there is a repulsive component to the potential which has the effect of pushing out the

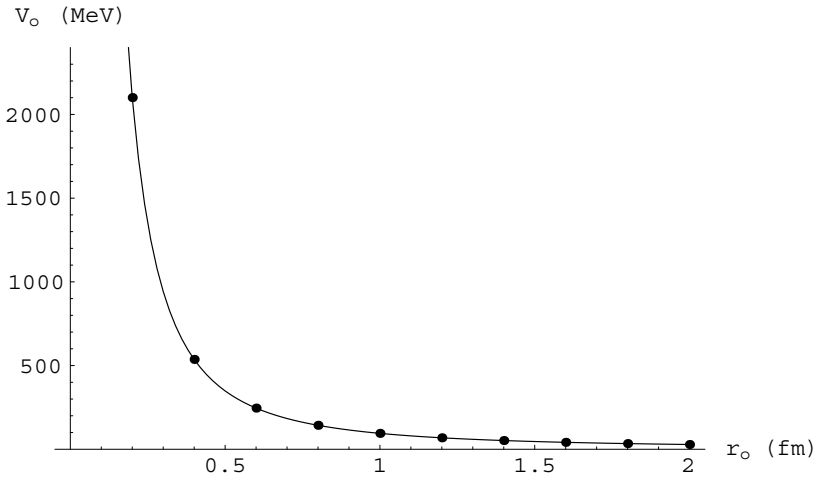

Figure 2. The range–depth relation obtained by using a variational approximation.

eigenfunctions as at the edge of an infinite well (compare with figure 1). By examining the plots of $\phi(p) = \mathrm{e}^{-p^2/m^2}\psi(p)$ (figure 4 shows one such plot for $r_0 = 1.43$ fm) where $\psi(p)$ are the eigenfunctions obtained from the quantum mechanics of smeared particles, we observe that $\phi(p)$ lies in $L^2(\mathrm{d}^3p)$. Therefore, the eigenfunctions obtained from the smeared particle analysis are normalizable with respect to $L^2(\mathrm{e}^{-2p^2/m^2}\,\mathrm{d}^3p)$. This motivates us to choose as our trial wavefunction

$$\psi(p) = \mathrm{e}^{p^2/m^2 - \alpha p r_0}. \tag{6.9}$$

The normalizability criterion in this measure ensures that

$$\mathrm{e}^{-p^2/m^2}\, p\psi(p) \to 0 \qquad \text{as} \quad p \to \infty \tag{6.10}$$

as required for bound states (and as is the case with our trial wavefunction). Furthermore, when the confines are large ($p^2/m^2 \ll 1$), $\psi(p)$ in equation (6.9) passes smoothly into the trial wavefunction we used when we applied ordinary quantum mechanics and which had yielded an accurate range–depth relation. Hence, our choice of the trial wavefunction is justified and, with the given volume element, we can determine the approximate range–depth relation that must hold if the potential function $V(p)$ is to give the value $E_0 = -2.226$ MeV for the binding energy. Numerical calculations performed in Mathematica reveal the range–depth relation shown in figure 5. The strength of the potential or depth of the well V_0' in figure 5 is lower than the strength of the potential V_0 obtained from ordinary quantum mechanics (figure 2) particularly for smaller values of r_0. The existence of a repulsive component to the potential which we have already observed from a plot of the eigenfunctions shown in figure 3 is verified. Moreover, the depth of the well V_0' in figure 5 is negative for $r_0 \leqslant 0.563$ fm. This implies the existence of a repulsive nucleon core with a radius $r_\mathrm{c} = 0.563$ fm, which is consistent with the phenomenologically obtained value of 0.6 fm.

Let us model the effective nucleon–nucleon interaction by a potential of the form

$$V(r) = -V_0 \frac{\mathrm{e}^{-r/r_0}}{r/r_0} + V_1 \frac{\mathrm{e}^{-r/r_1}}{r/r_1} \tag{6.11}$$

where $r_0 = 0.3596$ fm corresponding to σ meson exchange (attraction) and $r_1 = 0.2529$ fm corresponding to ω meson exchange (repulsion). This potential describes the main qualitative

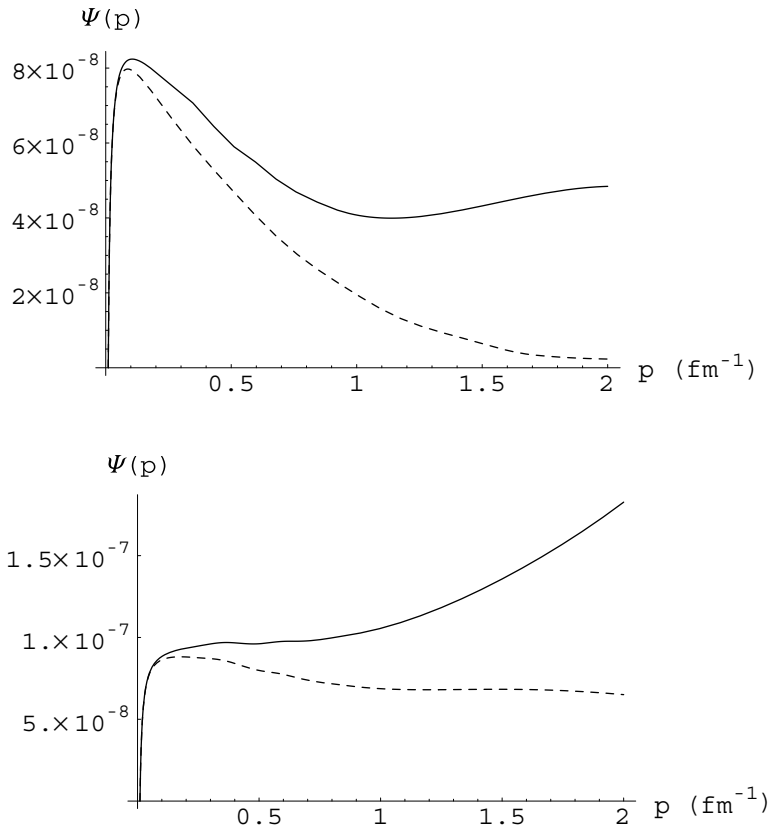

Figure 3. The solid curves show the eigenfunctions obtained from a smeared particle analysis. The top figure shows the eigenfunction at a range of $r_0 = 1.43$ fm and the bottom figure shows the eigenfunction at a range of $r_0 = 0.3596$ fm. For comparison the eigenfunctions obtained from ordinary quantum mechanics are also shown (dashed curves). The repulsion experienced by the nucleons, which is important at short distances, has the effect of pushing out the eigenfunctions.

features of the nucleon–nucleon interaction: a short-range repulsion between baryons coming from ω exchange and an intermediate-range attraction coming from σ exchange [5]. The repulsive component of the effective nucleon–nucleon interaction must be held accountable for the drop in the well depth from V_0 to V_0', which is observed at $r_0 = 0.3596$ fm. Since the ω exchange occurs at a range of $r_1 = 0.2529$ fm we require that

$$V(r = r_1) = -V_0' \frac{\mathrm{e}^{-r_1/r_0}}{r_1/r_0}. \tag{6.12}$$

The quantities $V_0 = 660.77$ MeV and $V_0' = -81.0$ MeV can be computed numerically or can be read from figures 2 and 5. A simple calculation yields the strength of the repulsive potential as $V_1 = 1419.07$ MeV. Figure 6 shows a plot of the effective nucleon–nucleon interaction. The potential is attractive at large distances and repulsive for small r. In terms of the coupling constants we can rewrite the effective nucleon–nucleon interaction as

$$V(r) = \frac{-g_\sigma^2}{4\pi} \frac{\mathrm{e}^{-r/r_0}}{r} + \frac{g_\omega^2}{4\pi} \frac{\mathrm{e}^{-r/r_1}}{r}. \tag{6.13}$$

Comparison with equation (6.11) yields $g_\sigma^2/4\pi = 1.20$ and $g_\omega^2/4\pi = 1.815$. Note that we are working in units with $\hbar = c = 1$. These theoretically obtained values of the coupling

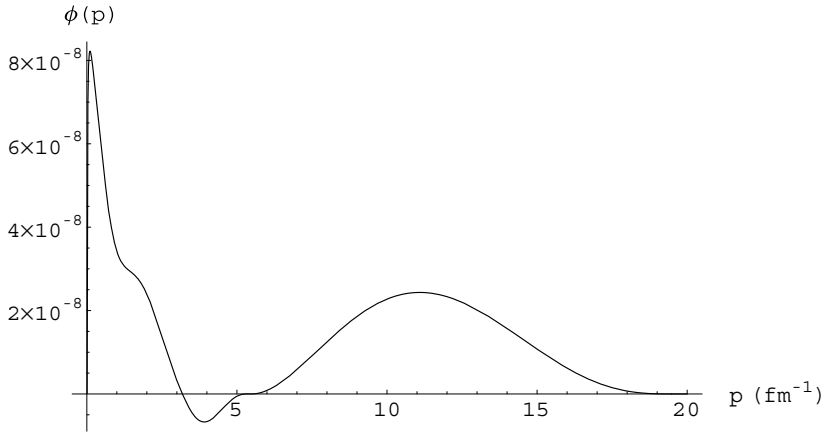

Figure 4. The plot of $\phi(p) = \mathrm{e}^{-p^2/m^2}\psi(p)$ where $\psi(p)$ is the wavefunction obtained from the quantum mechanics of smeared particles at $r_0 = 1.43$ fm.

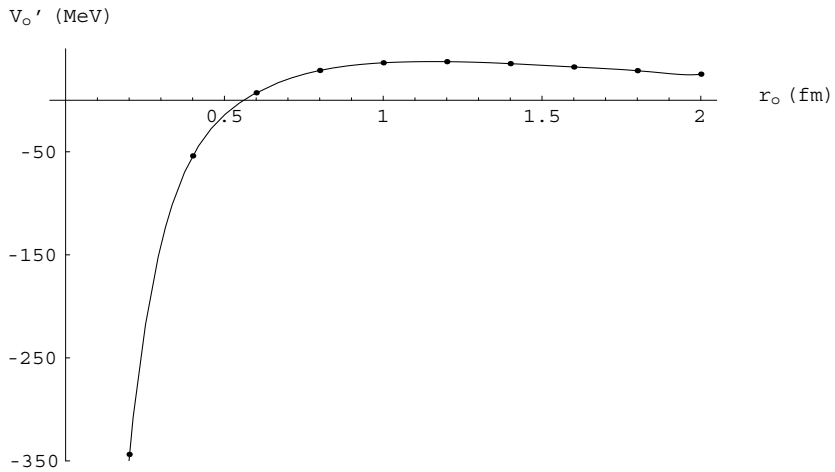

Figure 5. The range–depth relation obtained from a smeared particle analysis using the variational method. The strength of the potential is lowered, particularly for smaller values of r_0, indicating the existence of a repulsive component to the potential.

constants will differ from the phenomenological coupling constants because in our simple Yukawa model of the effective nucleon–nucleon interaction we have neglected important tensor interactions and spin–orbit terms which contribute to the form of the potential [3]. However, the ratio of the theoretical coupling constants $g_\omega^2/g_\sigma^2 = 1.512$, which compares the relative strength of the repulsive coupling and the attractive coupling, must be equal to the ratio of the phenomenologically determined coupling constants $g_{\omega_p}^2/g_{\sigma_p}^2$ in order for our simple Yukawa model to successfully describe the effective nucleon–nucleon interaction and to ensure the stability of the deuteron. Using the value $g_{\sigma_p}^2/4\pi = 7.303$ and multiplying by the ratio 1.512 we obtain the value of the phenomenological coupling constant of the ω meson as $g_{\omega_p}^2/4\pi = 11.03$. This value of the coupling constant differs by 1.85% from the value obtained from fitting the nucleon–nucleon scattering phase shifts and deuteron properties, which is equal to 10.83. Therefore, the quantum mechanics of smeared particles leads us to

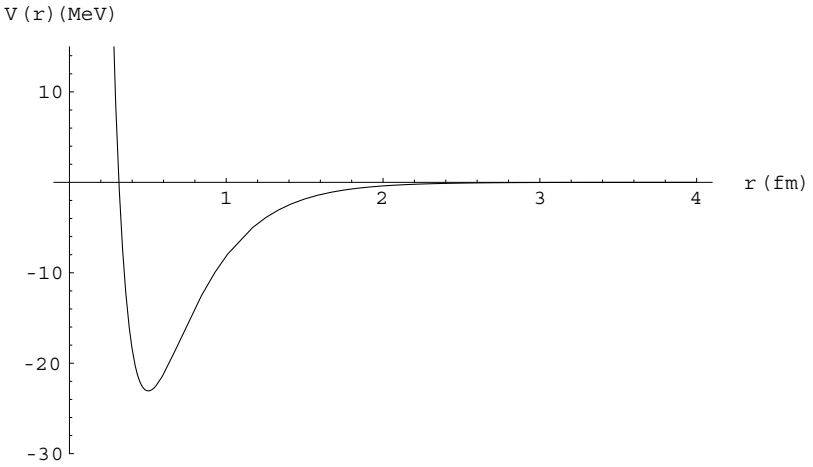

Figure 6. The effective nucleon–nucleon interaction.

values of the ω meson coupling constant and of the repulsive core radius which are consistent with the phenomenologically obtained values.

7. Conclusion

In this paper we have developed the Hilbert space representation theory of the quantum mechanics of smeared particles and demonstrated the invariance of such a mechanics under spatial translations and rotations. We have then applied this formalism to the smeared harmonic oscillator and the Yukawa potential. The results of the smeared harmonic oscillator are consistent with our classical intuition and, in the case of the Yukawa potential, we obtain accurate theoretical predictions of the hitherto phenomenologically obtained nucleon core radius and the ω meson coupling constant. In an age of increasing miniaturization, it is conceivable that as the confines of various quantum systems become comparable to the smearing of the confined particles, the quantum mechanics of smeared particles will play an important role in determining the dynamics.

Acknowledgments

The author would like to thank E C G Sudarshan and L Sadun for insightful discussions, and J R Klauder for his valuable comments. He would also like to thank R Zgadzaj for helping with the numerical calculations in Mathematica.

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